

# Maintaining Target Groundwater Levels Using Goal-Programming: Linear and Quadratic Methods

Amin Yazdanian, Richard C. Peralta

ASSOC. MEMBER  
ASAE

## ABSTRACT

**S**ustained-yield groundwater management strategies can be designed to closely maintain preassigned 'target' levels. Quadratic and linear goal-programming objective functions are used in two distinct models which minimize the sum of differences between 'target' and regionally optimized sets of groundwater levels. Constraints and bounds imposed on extractions, recharge and heads in each model assure that developed strategies are physically feasible and sustainable. The linear model is computationally more time-efficient, but numerical difficulties due to equality constraints are encountered when it is applied to large groundwater systems. The quadratic model requires less computer storage and is applied to the Grand Prairie of Arkansas as an example.

## INTRODUCTION

Groundwater management is virtually synonymous with attainment of certain potentiometric levels at certain points in time and space. In many instances, it is desirable and feasible to maintain a set of spatially distributed levels throughout time. A regional set of potentiometric levels to be maintained is usually determined on the basis of a number of local and regional factors of physical, social and economic nature.

The problem of determining a set of steady-state ground-water withdrawals that maintains a set of predetermined 'target' levels was addressed by Peralta and Peralta (1984). They used a linearized form of the Boussinesq equation (McWhorter and Sunada, 1977, p. 98) to develop such a set of withdrawal rates. They applied an iterative method to force the local steady-state rates to be physically feasible and realistic on both regional and local bases. No optimization criterion was included in their attempt, however. Yazdanian and Peralta (1986) demonstrated that an optimization process can be used to approximate the 'target' levels. In that effort, they used a quadratic goal-programming objective function to minimize the sum of squared deviations between an 'optimized' set of levels and the set of 'target' levels, subject to constraints and bounds on groundwater flow and on piezometric heads.

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The authors are: AMIN YAZDANIAN, Administrator, Maryland Agricultural Water Quality Cost-Share Program, Annapolis, MD; and RICHARD C. PERALTA, Associate Professor, Agricultural Engineering Dept., University of Arkansas, Fayetteville.

## OBJECTIVES

1. to extend the earlier modeling approach (Yazdanian and Peralta, 1986) by introducing an alternative linear formulation of the objective function,
  2. to compare the linear and quadratic formulations with regard to their computer time and memory requirements, and other merits and limitations,
  3. to demonstrate some features of the model by applying it to the Grand Prairie region of Arkansas.
- The following definitions are used in this paper. 'Pumping', expressed as a positive value, is the withdrawal of water from the aquifer through pumping wells. 'Sustained-yield' is an annual rate of withdrawal that maintains groundwater levels at specific elevations over a foreseeable period of time. 'Sustained-yield pumping strategy' is a specified pattern of spatially distributed pumping that maintains a specified potentiometric surface. 'Excitation' is any external stress on the aquifer system. A net negative excitation indicates recharge to , and a positive value indicates discharge from, the aquifer.

## THEORY

### Multi-Objective Optimization by Goal Programming

A goal-programming problem, in general form, may be expressed as:

$$\text{minimize } y = \sum_{k=1}^m |(h_{*k} - h_{tk})| x w_k \dots\dots\dots [1]$$

subject to:

$$h_{*k} \in F_h \quad k = 1, \dots, m \dots\dots\dots [2]$$

where:

- y = the value of the best compromise objective function
- m = total number of individual objectives
- $h_{*k}$  = the achieved value of the kth objective (initially unknown)
- $h_{tk}$  = target value (goal) of the kth objective (known)
- $w_k$  = a weighting factor
- $F_h$  = the set of feasible values for  $h_{*k}$
- $\in$  denotes 'element of'.

The weighting factors permit over-or under-emphasizing the achievement of individual objectives.

Equation [1] is non-linear because absolute values are used. A quadratic form that may be employed to minimize the sum of squared deviations can be expressed as:

$$\text{minimize } z = \sum_{k=1}^m \left\{ (h_{*k} - h_{tk}) \right\}^2 x w_k \dots\dots\dots [3]$$

subject to equation [2].

It is also possible to linearize equation [1] by replacing each difference term with two new variables:

$$|h_{*k} - h_{tk}| = d_k^+ + d_k^- \quad \dots \dots \dots [4]$$

where:

$d_k^+$  = the difference between achieved and target values of objective  $k$ , when more of this objective is achieved than required (over-achievement). Therefore, if the achieved value of  $k$  is more than its target value,  $d_k^+$  is positive valued—otherwise it is zero.

$d_k^-$  = the difference between target and achieved value of objective  $k$ , when less of this objective is achieved than needed (under-achievement). Therefore,  $d_k^-$ , which is also a positive quantity, exists only if the target value of  $k$  is greater than its achieved value—otherwise it is zero.

Both the target and achieved values of objective  $k$  are positive. Also, for any one objective, only one of these differences exists while the other is zero. In other words, one objective can not be both over-achieved and under-achieved.

Thus, the linear goal-programming problem is formulated as:

$$\text{minimize } y = \sum_{k=1}^m (d_k^+ + d_k^-) \times w_k \quad \dots \dots \dots [5]$$

subject to:

$$h_{*k} - d_k^+ + d_k^- = h_{tk} \quad k = 1, \dots, m \quad \dots \dots \dots [6]$$

$$h_{*k} \leq F_h \quad k = 1, \dots, m \quad \dots \dots \dots [2]$$

$$d_k^+, d_k^- \geq 0 \quad k = 1, \dots, m \quad \dots \dots \dots [7]$$

### Simulation of Groundwater Flow

A combination of Darcy's Law and the law of continuity produces the description of groundwater flow known as the Boussinesq equation. Expressed in terms of continuous partial derivatives for two-dimensional flow, the equation is:

$$\frac{\partial}{\partial x} \left( T \frac{\partial h}{\partial x} \right) + \frac{\partial}{\partial y} \left( T \frac{\partial h}{\partial y} \right) - Q = S \frac{\partial h}{\partial t} \quad \dots \dots \dots [8]$$

where  $T$  and  $S$  are the transmissivity and storage coefficients of the aquifer material, respectively,  $h$  is potential (or head) and  $t$  is time. In this equation, if there is no change in head with time, the term on the right hand side vanishes. In that case  $Q$  is the net steady-state excitation under constant hydraulic gradients.

Equation [8] is written in finite-difference form to describe steady-state flow in a heterogenous isotropic aquifer. For cell  $(i,j)$  in a two-dimensional system, the equation is:

$$q_{i,j} = (t_{i-1/2,j} (h_{i-1,j}) + (t_{i+1/2,j} (h_{i+1,j}) - (t_{i-1/2,j} + t_{i+1/2,j} + t_{i,j-1/2} + t_{i,j+1/2}) (h_{i,j}) + (t_{i,j-1/2} (h_{i,j-1}) + (t_{i,j+1/2} (h_{i,j+1}) \dots \dots \dots [9]$$

where:

$q_{i,j}$  = the net steady-state excitation in cell  $(i,j)$ ,  $L^3/T$

$h_{i,j}$  = the steady-state (static) head,  $L$

$t_{i-1/2,j}$  = the geometric average of the transmissivities of cells  $(i,j)$  and  $(i-1,j)$ ,  $L^2/T$

Expressed in matrix form for an  $n$ -cell system, equation [9] takes the form:

$$Q = [T] \{H\} \quad \dots \dots \dots [10]$$

where:

$\{Q\}$  is an  $(n \times 1)$  column vector of the net steady-state flow values,  $L^3/T$

$[T]$  is an  $(n \times n)$  symmetric banded matrix of finite-difference transmissivities,  $L^2/T$

and  $\{H\}$  is a column vector of static heads,  $L$ .

### MODEL FORMULATION

An aquifer system in this approach is represented by a square finite-difference grid of  $(n)$  cells. For appropriate treatment of each grid cell, a distinction is made between cells located on the periphery of the system (i.e., boundary cells) and the cells located inside the area (i.e., internal cells). Head and pumping are assumed to be variable in the  $(m)$  internal cells, while external recharge to these cells is assumed as being constant. No pumping is assumed to take place in the  $(n-m)$  boundary cells. Head is also assumed to remain constant in these cells while recharge from outside the system is treated as a variable. The condition shown by equation [2] is expressed via equation [10] for the groundwater system, with  $Q$  and  $H$  representing the involved variables.

### Quadratic Formulation

In this formulation the objective function given by equation [3] is minimized subject to constraints on pumping and recharge and bounds on heads in each finite difference grid square. Expressed in standard quadratic programming form, in matrix notation, the problem is given as:

$$\text{minimize } z = -2(H_{tw}) \{H_*\} + (1/2) (H_*)^T [W] \{H_*\} + (H_{tw}) \{H_t\} \quad \dots \dots \dots [11]$$

subject to:

$$\{L_q\} \leq \{Q\} = [T] \{H\} \leq \{U_q\} \quad \dots \dots \dots [12]$$

$$\{L_h\} \leq \{H_*\} \leq \{U_h\} \quad \dots \dots \dots [13]$$

where:

- $z$  = the value of the objective function,  $L^2$
- $\{H_{tw}\}$  = a  $1 \times m$  row vector whose elements are the product of the known target heads in the internal cells and the weighting factors,  $L$
- $\{H_*\}$  = the  $m \times 1$  column vector of initially unknown heads in the internal cells that are optimized,  $L$
- $\{H_*\}^T$  = the transpose of column vector  $\{H_*\}$ ,  $L$
- $[W]$  = the  $m \times m$  diagonal matrix whose diagonal elements are two times the weighting factors, (dimensionless)
- $\{H_t\}$  = the  $m \times 1$  column vector of the known target heads in the internal cells,  $L$
- $\{L_q\}$  = an  $n \times 1$  column vector whose elements are the lower bound on pumping (or recharge) in all the cells in the system,  $L^3/T$
- $\{U_q\}$  = an  $n \times 1$  column vector whose elements are the upper bound on pumping (or recharge) in all the cells in the system,  $L^3/T$
- $\{Q\}$  = an  $n \times 1$  column vector of net steady-state pumping (or recharge) rates for all the cells, ( $L^3/T$ ). Using equation [10],  $\{Q\}$  is expressed in terms of transmissivity  $[T]$  and head  $\{H\}$ . The column vector  $\{H\}$  of heads for the entire system is different from the vector of heads for the internal cells,  $\{H_*\}$ , only in that it contains the heads for both the boundary and internal cells
- $\{L_h\}$  = an  $m \times 1$  column vector of lower bounds on optimal steady-state heads in the internal cells,  $L$
- $\{U_h\}$  = an  $m \times 1$  column vector of upper bounds on optimal steady-state heads in the internal cells,  $L$

The first and second terms in equation [11] are linear and quadratic, respectively, in terms of the unknown heads. The third term consists of constants.

In order to reduce the number of constraints required for modeling equation [12] the constraints are formulated as:

$$[T] \{H\} \geq \{L_q\} \dots \dots \dots [14]$$

That is, the net steady-state excitations must be greater than or equal their assigned lower limits. Then, to assure that  $\{Q\}$  also remains less than or equal to  $\{U_q\}$ , as required by equation [12], the slack variables associated with constraints [14] are bounded as:

$$\{0\} \leq \{X\} \leq \{U_q\} - \{L_q\} \dots \dots \dots [15]$$

in which  $\{X\}$  is an  $n \times 1$  vector of slack variables applied to constraints (14) during optimization to convert them to equality constraints. Elements of  $\{X\}$  have dimensions of  $L^3/T$ . By imposing the bounds shown by equation [15], whenever an element of  $\{X\}$  is equal to zero, the corresponding element of  $\{Q\} = [T]\{H\}$  is at its lower bound. In contrast, when an element of  $\{X\}$  is equal to  $\{U_q\} - \{L_q\}$ , the corresponding element of  $\{Q\}$  is at its upper bound, because in this case from

$$[T]\{H\} - \{X\} = \{L_q\}$$

we have

$$\{Q\} - \{U_q\} + \{L_q\} = \{L_q\} \text{ or } \{Q\} = \{U_q\}$$

The number of constraints required for modeling [14] is half of that required for modeling equation [12]. In the quadratic formulation, then, for a finite-difference grid with a total of  $(n)$  grid squares, and  $(m)$  internal cells, there are  $(m)$  variable heads and  $(n)$  slack variables associated with the  $(n)$  constraints. Therefore, there is a total of  $(n+m)$  variables, all of which are bounded. The total number of initial constraints, all inequality, is  $(n)$ .

### Linear Formulation

In this approach the objective function given by equation [5] is optimized subject to constraints expressed by equations [6] and [12], and bounds expressed by equations [7] and [13]. In matrix notation, the complete formulation is :

$$\text{minimize } y = (V) \{D^+\} + (V) \{D^-\} \dots \dots \dots [16]$$

subject to :

$$\{H_*\} - \{D^+\} + \{D^-\} = \{H_t\} \dots \dots \dots [17]$$

$$\{L_q\} \leq \{Q\} = [T] \{H\} \leq \{U_q\} \dots \dots \dots [12]$$

$$\{L_h\} \leq \{S_*\} \leq \{U_h\} \dots \dots \dots [13]$$

$$\{D^+\}, \{D^-\} \geq 0 \dots \dots \dots [16]$$

where:

- $y$  = the value of the objective function,  $L$
- $(V)$  = the  $1 \times m$  row vector of weighting factors, (dimensionless)

$\{D^+\}$  and  $\{D^-\}$  are each an  $m \times 1$  column vector of over- and under-achievements, respectively,  $L$

$\{H_*\}$ ,  $\{H_t\}$ ,  $\{L_q\}$ ,  $\{U_q\}$ ,  $\{Q\}$ ,  $[T]$ ,  $\{H\}$ ,  $\{L_h\}$ , and  $\{U_h\}$  are the same as given for the quadratic formulation.

Assuming a system with  $(m)$  internal cells and  $(n)$  total cells, there will be  $(m)$  variable heads, overachievements, and underachievements. There are also  $(n)$  slack variables associated with the inequality constraints expressed by equation [12]. Therefore, there is a total of  $(n+3m)$  variables. All variables are bounded. The constraints consist of  $(m)$  equality and  $(n)$  inequality constraints, a total of  $(n+m)$ .

In computer coding of both the linear and quadratic models described above, constraints on recharge are formulated in a separate subroutine. The call to that subroutine is optional. When optimization is performed without constraining the recharge, another subroutine uses equation [9] to compute the recharge rates needed at the boundary cells to support the optimal elevations.

A subroutine developed by Leifsson et al (1981) is used for optimization. This subroutine has options for optimization of linear and convex quadratic objective functions. When applying the model, the method of minors is used to verify the convexity of the quadratic

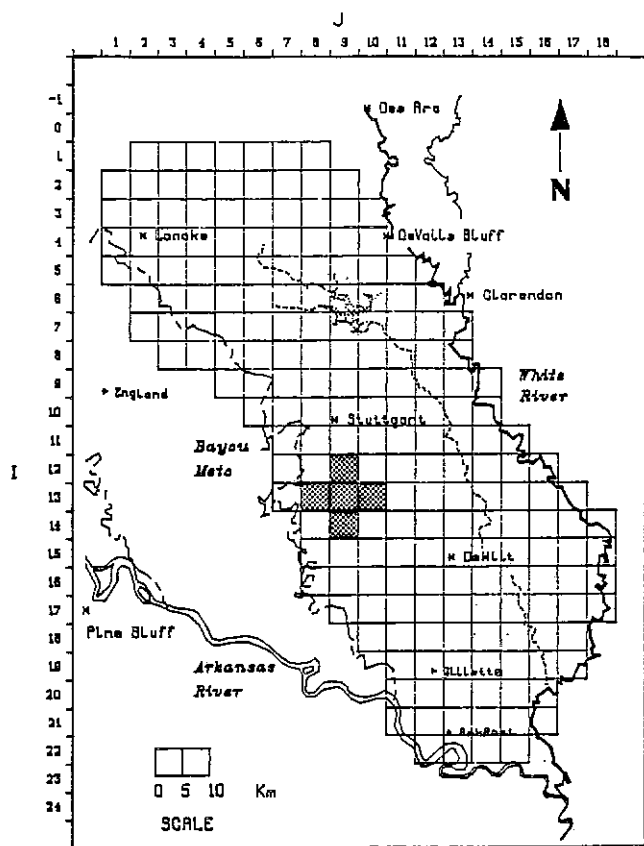


Fig. 1—Grand Prairie study area.

objective function, equation [11]. Since concave linear constraints are used, global optimality is assured for both linear and quadratic models.

## RESULTS

The linear and quadratic models have similar features and can be used for the same management purposes, including:

- Estimation of realistic steady-state groundwater elevations that most closely approximate a predetermined set of target levels.
- Development of more or less uniform regional sustained-yield pumping strategies, by changing lower bounds on pumping.
- Development of strategies for attainment of exact target gradients at specific locations within a regional plan.
- Estimation of the steady-state recharge needed to maintain specific target elevations most closely.

These features are demonstrated, in subsequent sections, using the quadratic model to develop strategies for the entire Grand Prairie region of Arkansas (Fig. 1). Before doing so, however, a brief description of the aquifer physical properties is provided and the results obtained by applying both programs to subareas of the region are compared.

The sources of recharge to the Grand Prairie study area (Fig. 1) are known to be the hydraulic connections along its boundaries (Peralta et al., 1985b). That is, no direct recharge by deep percolation takes place. This is due to a dense clay cap that covers the aquifer formation. There are no stream/aquifer connections inside the study area either.

In its unstressed state the aquifer was confined

throughout the study area. At the present time it is confined in the springtime, before the pumping season, only along the western, southern and eastern boundaries. The degree of confinement is such that the aquifer may be considered to be unconfined in the vicinity of pumping wells.

Physical properties of the Quaternary aquifer underlying the Grand Prairie have been estimated and verified by several investigators. Engler et al. (1945) conducted pumping tests and reported an aquifer storage coefficient of 0.3 and a hydraulic conductivity of 77.4 m/day (254 ft/d). Sniegocki (1964) reported a storage coefficient of 0.3 and a hydraulic conductivity of 81.4 m/d (267 ft/d). Griffiths (1972) developed and validated a two-dimensional groundwater simulation model of the same aquifer. He used a storage coefficient of 0.3 and a hydraulic conductivity of 81.4 m/d (267 ft/d). Broom and Lyford (1981) modeled an adjacent part of the same aquifer and obtained best results when using a storage coefficient of 0.3 and a hydraulic conductivity of 82.3 m/d (270 ft/d). While validating a two-dimensional groundwater simulation model (Verdin et al., 1981), Peralta et al. (1985b) found the best fit between historic and simulated responses using a storage coefficient of 0.3 and a hydraulic conductivity of 82.3 m/d (270 ft/d). A hydraulic conductivity of 82.3 m/d was used in calculation of transmissivities in this study.

## Comparative CPU Time-efficiency of the Linear and Quadratic Models

To compare execution times, each model was tested with three groundwater systems of different sizes. These systems included parts of the Grand Prairie study area. The specifics of the three systems and the total CPU time required for execution of each are shown in Table 1.

The three factors considered in this comparison are type of problem (LP or QP), number of variables and number of constraints. All other factors affecting the execution time, such as computational accuracy criteria, were the same for both models. The results show that although increasing the number of variables and constraints rapidly increases the CPU time for both types, the increase is much faster for the quadratic formulation. Therefore, for problems of the size range tested here, the linear formulation is significantly more time-efficient than the quadratic formulation.

However, for the same number of finite-difference grid-points the linear formulation always involves more variables and constraints, therefore, it needs more computer storage. In particular, the linear model uses an additional equality constraint (equation [17]) for each

TABLE 1. COMPARISON OF COMPUTER TIME-EFFICIENCY OF THE LINEAR AND QUADRATIC OPTIMIZATION MODELS

| System no.         |          | 1   | 2    | 3     |
|--------------------|----------|-----|------|-------|
| No. of cells       | Internal | 16  | 38   | 59    |
|                    | Total    | 32  | 62   | 90    |
| No. of variables   | LP       | 80  | 176  | 267   |
|                    | QP       | 48  | 100  | 149   |
| No. of constraints | LP       | 48  | 100  | 149   |
|                    | QP       | 32  | 62   | 90    |
| CPU time, s        | LP       | 1.7 | 5.4  | 28.3  |
|                    | QP       | 2.4 | 22.0 | 104.6 |

internal cell. Unfortunately, when optimizing using the linear formulation for the 152-cell Grand Prairie, a number of these constraints were violated. Numerical difficulties in dealing with equality constraints have been also reported by Elango and Rouve (1980), Evans and Remson (1982) and Gorelick (1983). These difficulties dictated the use of the quadratic form when modeling the entire Grand Prairie study area.

The largest size problem successfully optimized using the linear model was the 90-cell system shown in Table 1. That system required 59 equality constraints. On the other hand, as an objective function within the SSTAR (Steady-State TARget) model, the quadratic form has been successfully applied to a study area of 376 cells (Peralta et al. 1985a).

The University of Arkansas computing system consists of an AMDAHL 470-V6 main processor, considered equivalent to an IBM 370. The models presented are coded in VS/FORTRAN. The CPU time required to compile either of the two programs for linear or quadratic optimization is about 8.0 s.

### Comparison of Sample Results from the Linear and Quadratic Models

Sample target groundwater elevations, for a system consisting of 90 finite-difference cells, are shown in Fig. 2. The area covers a central part of the study area shown in Fig. 1. It extends from I=6 to I=17 and from J=6 to J=14. Hydrologic assumptions adopted for modeling this system are based on the data compiled for the entire study. It is assumed that recharge to this system can only take place through the boundary cells. That is, there is no external recharge to any of the 59 internal cells. The boundary cells are assumed to maintain the constant-head elevations shown in Fig. 2. Constraints on recharge at the boundary cells were set at arbitrarily high rates. The minimum pumping in the internal cells was assumed to be zero, and arbitrarily

|    |    |    |    |    |    |    |    |
|----|----|----|----|----|----|----|----|
|    | 35 | 34 | 34 | 38 | 42 | 45 |    |
| 39 | 37 | 35 | 34 | 36 | 40 | 43 | 46 |
| 41 | 38 | 35 | 33 | 34 | 37 | 41 | 45 |
| 44 | 41 | 36 | 32 | 33 | 36 | 39 | 46 |
| 46 | 43 | 39 | 35 | 34 | 34 | 37 | 42 |
|    | 44 | 40 | 36 | 34 | 33 | 35 | 39 |
|    | 46 | 41 | 36 | 33 | 31 | 33 | 39 |
|    | 46 | 42 | 37 | 32 | 30 | 30 | 36 |
|    | 44 | 38 | 32 | 30 | 29 | 31 | 34 |
|    | 47 | 41 | 34 | 30 | 29 | 30 | 33 |
|    | 50 | 45 | 37 | 33 | 30 | 30 | 33 |
|    |    | 48 | 42 | 37 | 33 | 31 |    |

Fig. 2—Target elevations in a 90-cell system tested with both linear and quadratic models, (meters above mean sea level).

|   |    |   |   |   |   |   |    |
|---|----|---|---|---|---|---|----|
|   | *  | * | * | * | * | * |    |
| * | 0  | 0 | 0 | 0 | 0 | 0 | *  |
| * | 0  | 0 | 0 | 0 | 0 | 0 | *  |
| * | 0  | 1 | 0 | 0 | 0 | 1 | 0  |
| * | -1 | 0 | 0 | 0 | 0 | 1 | -1 |
| * | 0  | 0 | 0 | 0 | 0 | 1 | 0  |
| * | 0  | 0 | 0 | 0 | 0 | 0 | 0  |
| * | 0  | 0 | 0 | 0 | 0 | 0 | 0  |
|   | *  | 0 | 0 | 0 | 0 | 0 | *  |
|   | *  | 0 | 0 | 2 | 2 | 0 | *  |
|   | *  | 0 | 2 | 1 | 1 | 0 | *  |
|   |    | * | * | * | * | * |    |

(a)

|   |    |    |   |   |    |   |    |
|---|----|----|---|---|----|---|----|
|   | *  | *  | * | * | *  | * |    |
| * | 0  | 0  | 0 | 0 | 0  | 0 | *  |
| * | 0  | -1 | 0 | 0 | 0  | 0 | *  |
| * | 0  | 0  | 0 | 1 | -1 | 1 | 0  |
| * | -1 | -1 | 0 | 0 | 1  | 1 | -1 |
| * | 0  | 0  | 0 | 0 | 0  | 1 | 0  |
| * | 0  | 0  | 0 | 0 | 0  | 0 | 0  |
| * | 0  | 0  | 0 | 0 | 0  | 0 | 1  |
|   | *  | 0  | 0 | 0 | 0  | 1 | *  |
|   | *  | 0  | 0 | 0 | 0  | 1 | *  |
|   | *  | 0  | 0 | 0 | 0  | 1 | *  |
|   | *  | 0  | 0 | 0 | 0  | 1 | *  |
|   |    | *  | * | * | *  | * |    |

(b)

Fig. 3—Difference between target elevations of Fig. 2 and optimal elevations developed by a) the linear model, b) the quadratic model, (meters).

high maximum pumping rates were allowed in those cells.

Two optimal sets of elevations were obtained using the linear and quadratic models. The differences between each optimal elevation set and the target elevations are shown in Fig. 3. The regional sum of absolute differences shown in Fig. 3 for the linear model is 12 m (39 ft) while that for the quadratic model is 15 m (49 ft). Nonetheless, the maximum local difference for the linear model is 2 m while that for the quadratic model is 1. Apparently the quadratic model approximated the target elevations more smoothly. This is expected since the quadratic model minimizes the sum of squares of differences between target and optimal values while the linear model minimizes the sum of differences themselves. A strategy developed using the quadratic model should have fewer large differences between target and optimal values at individual cells than the linear model. Therefore, the quadratic model may be preferred if large local differences between target and optimal levels are to be avoided without intensive use of restrictive bounds on water levels.

For the above example, the total regional steady-state pumping from the quadratic model was 56.9 million cubic meters (Mm<sup>3</sup>) per year while that from linear model was 54.6, a difference of 4.2%. From a practical perspective, it should be mentioned that there are several thousand wells in the Grand Prairie. By setting {U<sub>q</sub>} equal to the capacity of the wells in each internal cell, it can be assumed that the wells in each cell are capable of pumping at the optimal rate determined by the models. It is also assumed that the spatial distribution of those wells will not cause water levels to significantly deviate from the optimal levels.

### Sustained-Yield Strategies with Constrained Recharge

A basin-wide management objective may be to

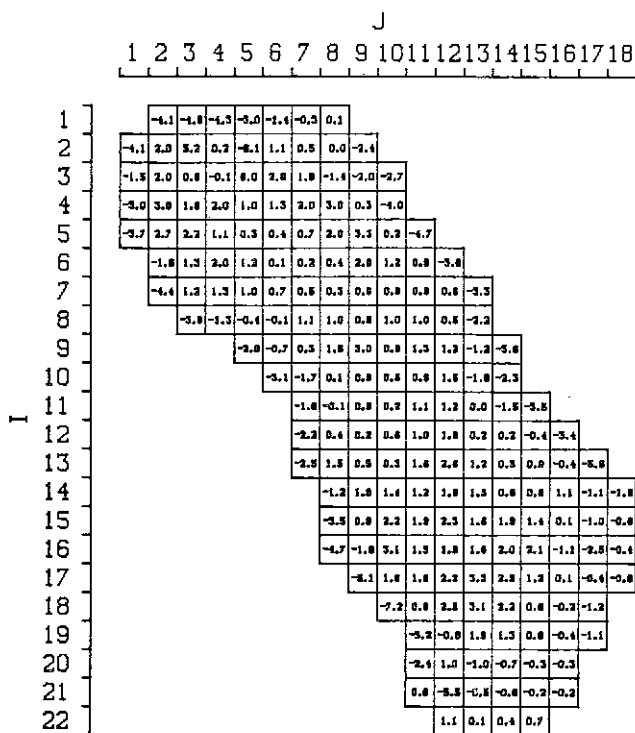


Fig. 4—Steady-state pumping (and recharge) computed for kriged 1983 springtime elevations, (MCM), using Equation (9).

maintain a groundwater potentiometric level as close as possible to 'the currently existing level'. Since, in reality, groundwater elevations fluctuate during the year, 'current level' is not a static level. It must be approximated by a static level to be associated with a steady-state pumping strategy.

Two sets of potentiometric elevations, derived from groundwater elevations observed in the study area (Fig. 1) in the springs of 1982 and 1983, were used as 'target' elevations to develop two pumping strategies. These strategies are referred to as Strategy 1 and 2, respectively. The statistical procedure known as kriging (Sophocleous et al., 1982) was used to interpolate the elevations from observation wells to the center of each cell.

The steady-state excitations associated with the 1983 'target' elevations are calculated, using equation [9] (Fig. 4). The positive and negative values in Fig. 4 indicate discharge (or pumping) and recharge, respectively. Since, as mentioned earlier, no recharge sources exist inside the Grand Prairie area, the negative values (steady-state recharges) shown in the internal cells cannot exist in reality. Therefore, in support of a previous statement, the potentiometric surface associated with such an excitation pattern is not a static surface.

To develop the static springtime potentiometric surfaces and their associated annual pumping strategies, using the management model, bounds were imposed on all variables. As explained in the model formulation, variables in the model consist of heads and pumping in internal cells and recharge in boundary cells. The following bounds were imposed on variables in the development of Strategies 1 and 2.

- The upper bound on optimal steady-state head in each internal cell was equal to the ground surface elevation in that cell.

- The elevation of the base of the aquifer in each internal cell was the lower bound on steady-state head in that cell.

- The upper bounds on pumping for Strategy 1 were the 1982 pumping values estimated by Peralta et al. (1985b). Maximum potential water needs used as upper bound on pumping in internal cells for Strategy 2. Maximum potential need in each cell was estimated by Ranjha et al. (1985) using land suitability for rice or irrigated soybean and wheat, cropped under average climatic conditions.

- Lower bounds on pumping equal to ten percent of the imposed upper bounds were applied for Strategy 1. The lower bounds for Strategy 2 were set to zero. It will be shown that the lower bound on pumping has a significant bearing on the areal distribution of the resulting optimal steady-state pumping.

The two strategies presented in this section were developed with constraints imposed on the recharges that could take place at the boundary cells. Upper limits on annual recharge to the constant-head boundary cells for Strategy 1 were the average of the values calculated to have occurred between 1972 and 1982, based on observed springtime gradients. The maximum recharge rates estimated to have occurred at those cells for the period 1972 to 1983 were used for development of Strategy 2.

Lower limits on recharge at the boundary cells were relaxed in the sense that large positive values were assigned to them. Since the regional sustained yield equals the net regional sum of all discharges and recharges, it may be necessary to permit the aquifer to

TABLE 2. BOUNDS IMPOSED ON VARIABLES FOR DEVELOPMENT OF OPTIMAL STEADY-STATE PUMPING STRATEGIES

| Strategy | Variable | Limit      | Description  |
|----------|----------|------------|--|
| 1 :      | Head     | Upper      | Ground surface elevation.  |
|          |          | Lower      | Elevation of the aquifer base.   |
|          | Pumping  | Upper      | 1982 pumping.  |
|          |          | Lower      | 10% of the upper limit.  |
|          | Recharge | Upper      | 1972-82 average recharge calculated from 'observed' gradients.                       |
|          |          | Lower      | (Relaxed).   |
| 2 :      | Head     | Upper      | Same as Strategy 1.  |
|          |          | Lower      | Same as Strategy 1.  |
|          | Pumping  | Upper      | Pumping estimated based on soil cropping potentials and average climatic conditions. |
|          |          | Lower      | Zero.  |
|          | Recharge | Upper      | 1972-83 maximum recharge calculated from 'observed' gradients.                       |
|          |          | Lower      | (Relaxed).   |
| 3 :      | Head     | Upper      | Same as Strategy 1.  |
|          |          | Lower      | Same as Strategy 1.  |
|          | Pumping  | Upper      | Same as Strategy 2.  |
|          |          | Lower      | Zero.  |
|          | Recharge | No limits. |  |

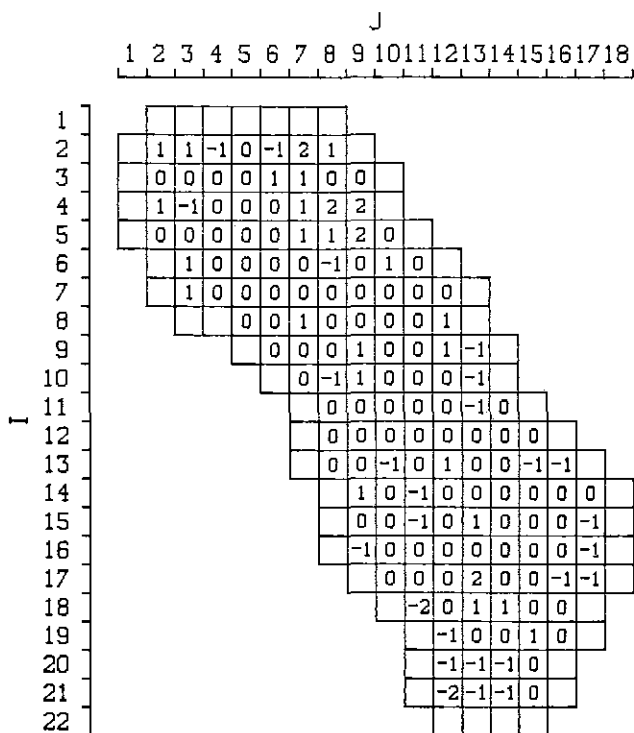


Fig. 5—Difference between optimal and target elevations of Strategy 1, (meters).

discharge at some locations so that it can recharge at others.

Table 2 provides a quick reference for the bounds imposed on head, pumping and recharge variables in developing Strategies 1 and 2. It also contains the same information for another strategy that is presented in a following section.

Fig. 5 shows the difference between optimum and target elevations for Strategy 1. The largest difference between the two levels was 2.4 m (8 f), observed in cell

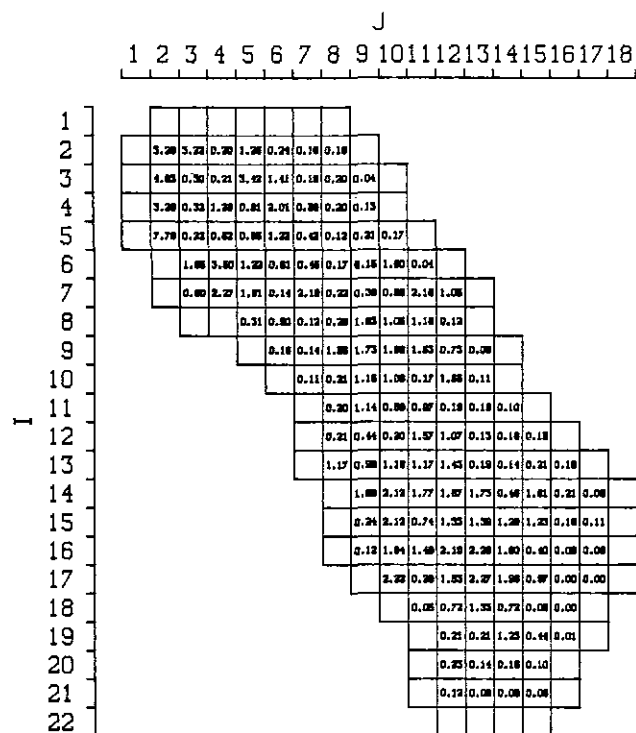


Fig. 6—Strategy 1 annual optimal pumping rates, (Mm³).

(5,9), (Rounded to 2 m in Fig. 5).

As stated above, for Strategy 1 the optimal pumping was forced in each cell to be greater than or equal to 10% of the 1982 pumping. Fig. 6 displays the spatial distribution of steady-state pumping corresponding to the optimum elevations. The total optimum steady-state pumping is 143 Mm³ (116,000 ac-ft). This is 41% of the 1982 season actual pumping of 349 Mm³ (283,000 ac-ft). Engler et al. (1945) applied Darcy's Law to the average springtime gradients along the boundary of the Grand Prairie area to estimate a regional sustainable pumping. They estimated an annual sustained yield of 148 Mm³ (120,000 ac-ft), which is very close to the value estimated by the methodology of Strategy 1. A main advantage of the approach presented here is that it calculates a sustainable spatial distribution of pumping. Furthermore, by appropriately selecting the bounds on pumping in each cell, this pumping strategy can be compatible with the present pattern of withdrawals.

Strategy 2, developed using a lower bound of zero on optimal pumping, produced a slightly higher regional sustained yield than Strategy 1. This was expected because Strategy 1 was developed with a more restricted pumping decision space. Strategy 2 produced a regional sustained yield of 145 Mm³ (117,500 ac-ft).

The spatial distribution of optimal pumping for Strategy 2 is different from that of Strategy 1. In Strategy 2 there are many cells in which no pumping is allowed. Fig. 7, displaying the spatially distributed pumping for Strategy 2, may be compared with Fig. 6 to contrast the areal distribution of pumping for the two strategies. Note that by changing the lower bound on optimal pumping, it is possible to develop strategies that give various degrees of spatial uniformity in sustained yield pumping rates. A more evenly distributed pumping may be socially more implementable. It is obvious, however, that as the range of acceptable values for optimal pumping is narrowed by raising the lower bound, the solution space

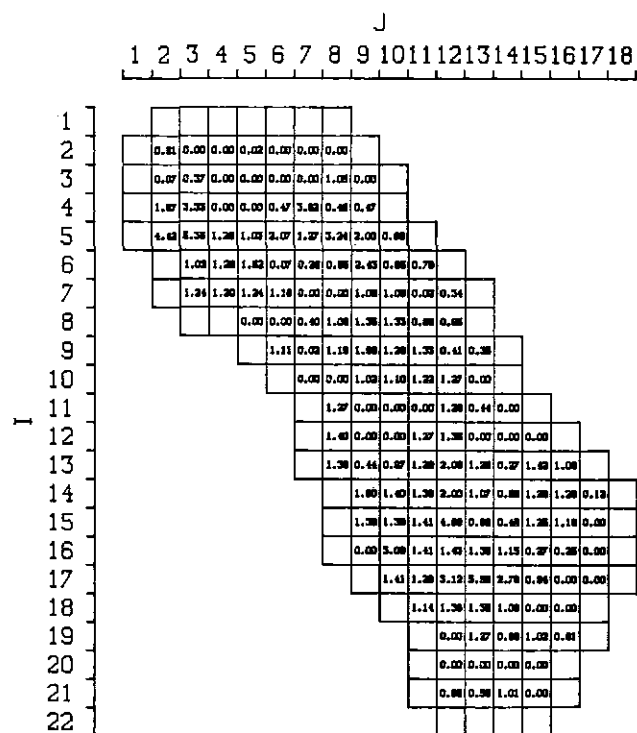


Fig. 7—Strategy 2 annual optimal pumping rates, (Mm³).

becomes more restricted. This is initially reflected in the optimal solution as an increased regional sum of differences between optimal and target elevations. Finally, if the minimum acceptable pumping is too high, finding a solution to the problem becomes impossible. For the Grand Prairie study area and using the constraining conditions imposed for Strategy 1, a lower bound on pumping equal to about 14 percent of the upper bound is the limit beyond which no feasible solution can be found.

The pumping strategies presented and discussed above consist of an annual amount of withdrawal in each cell based on the optimized springtime groundwater elevations. Though they have been referred to as sustained-yield pumping strategies already, in actuality the following conditions must be satisfied before they can be considered as such:

1. The physical boundary conditions used for their development must exist. In other words, the recharge assumed to occur from extensions of the aquifer outside the area to a boundary (constant-head) cell must be available when the groundwater level of the constant-head cell is at its specified elevation.

2. Since groundwater withdrawal varies through the year, with intensive pumping concentrated in the summer months for irrigation, a dynamic situation exists in reality. Under this condition, there must be an assurance that groundwater elevations return to their spring levels year after year.

To verify the latter point, the annual pumping volumes in Fig. 6 were divided into appropriate monthly values with pumping concentrated only in the irrigation season. A simulation was then performed using a model validated and applied to the Grand Prairie Quaternary aquifer by Peralta et al. (1985b). After 120 simulated months starting at the optimal elevations in March, the last 12 months of the simulated elevations were compared with the initial (optimal) elevations. The largest difference was observed in cell (13,9) in August and September where groundwater elevations were 0.35 m (1.2 ft) lower than the optimal elevations. The differences for the spring months were even less significant. Therefore, implementation of the strategies maintains the optimal elevations. Since the recharge constraints were selected to represent historic recharge rates, the first condition mentioned above is also satisfied, and the presented pumping strategies are sustainable.

#### Attainment of Specific Gradients in A Subarea

As mentioned earlier, the management model can be applied to attain specific local groundwater gradients and pumping rates in a subarea within a regional plan. An example of such a local management objective is presented in this section and the model application to achieve that objective is demonstrated.

Peralta et al. (1986) identified a subarea of the Grand Prairie where assuring a minimum saturated thickness in cell (13,9) and groundwater gradients across that cell were critical factors for availability of groundwater for drought protection. Specific target elevations required in cell (13,9) and its four neighboring cells (shaded cells in Fig. 1) to provide the necessary gradients and an annual pumping rate of 764,000 m<sup>3</sup> (619 ac-ft) are shown in column 2 of Table 3.

TABLE 3. GROUNDWATER ELEVATIONS IN THE CRITICAL REGION (METERS ABOVE MEAN SEA LEVEL)

| Cell<br>i,j | Elevations                            |                  |                   |                  |
|-------------|---------------------------------------|------------------|-------------------|------------------|
|             | Target (for<br>drought<br>protection) | First<br>optimum | Second<br>optimum | Third<br>optimum |
| (1)         | (2)                                   | (3)              | (4)               | (5)              |
| 13,9        | 36.3                                  | 36.0             | 36.3              | 36.3             |
| 13,8        | 42.4                                  | 42.1             | 42.4              | 42.4             |
| 13,10       | 31.7                                  | 30.5             | 31.7              | 31.7             |
| 14,9        | 38.1                                  | 38.4             | 38.4              | 38.1             |
| 12,9        | 35.3                                  | 35.0             | 35.3              | 35.3             |

Elevations shown in column 2 of Table 3 were used as target elevations to develop a regional strategy. Except for the modified target elevations in the critical subarea, all other input were the same as used for development of Strategy 1, including bounds on variables. The optimum levels (i.e. regionally 'best possible' levels with the least sum of deviations from the target levels) are shown in column 3 of Table 3, denoted as the first optimum. It can be observed that the exact target elevations in the critical region, and therefore the necessary gradients, were not achieved within the regional plan. Additional constraining conditions were required to achieve the precise desired elevations and gradients.

A second optimization was then performed, using the optimal elevations developed for this first optimization as an initial feasible solution. High weighting factors were assigned to the five cells of the critical region. The difference observed between the target and the first optimal elevation was used to select the magnitude of the weighting factor applied to each of the five cells. The weighting factor assigned to cell (13,10) was, therefore, about four times as large as those assigned to the other four cells. The weighting factors assigned to the latter cells were about 25 times greater than the average weighting factors for cells other than the five cells in the region. The average regional weighting factor was 0.10.

The optimal elevations developed in the second solution are shown in column 4, Table 3. In this solution, target elevations were achieved in four cells but cell (14,9) with an elevation of 38.4 m, remained 0.3 m (1 ft) too high. This second solution essentially overachieves the requirements in the critical cell (13,9) by providing a higher gradient towards that cell than was needed for drought protection. In addition, the corresponding sustainable annual pumping rate in cell (13,9) under this solution is 772,000 m<sup>3</sup> (626 ac-ft), slightly more than is required.

To achieve the desired elevations with a certain accuracy (e.g., within 0.1 m), a third optimization was performed in which the second solution was used as the initial feasible solution. The following bounds were imposed to the four cells that had already achieved their targets:

$$36.25 < h_{13,9} < 36.35$$

$$42.35 < h_{13,8} < 42.45$$

$$31.65 < h_{13,10} < 31.75$$

$$35.25 < h_{12,9} < 35.35$$

where  $h_{i,j}$  denotes the optimal groundwater elevation in cell (i,j). The weighting factor for cell (14,9) was then raised to 40 times that used in the second solution, and



another optimization was performed.

The final optimal elevations are shown in column 5 of Table 3. They are equal to the target elevations in those five cells. This final strategy meets the requirements of the drought protection plan. The total regional sustained-yield pumping was virtually the same as for Strategy 1 (within 0.1 percent), i.e., 143 Mm<sup>3</sup> (116,000 ac-ft).

### Estimation of Recharges That Support Optimal Static Elevations

An application of the optimization methodology described herein is to estimate the recharge required, at the physically recognized recharge locations, to support a set of truly static elevations which is regionally the closest possible to a set of target elevations. This is demonstrated in this section.

As mentioned previously, the observed springtime elevations are not static levels. Application of the optimization model permits development of feasible steady-state levels by imposing constraints on physically impossible conditions, such as recharge in the internal cells in the Grand Prairie. At the same time the model permits the constraints on recharge at the boundary cells to be removed. The regional groundwater surface developed in this case is a static surface, including only the internal (or variable-head) cells, which is closest to the input target surface (e.g., a set of springtime elevations that may not be static). Equation [9] is, then, used through the model to estimate the recharge required at the boundary cells to support the static groundwater level developed for the internal cells.

Strategy 3, presented here as an example, is developed without recharge constraints at the boundary cells. Except for the constraints on optimal recharge, Strategy 3 is developed using the same data and criteria as Strategy 2 (see Table 2).

The recharges that will occur at the boundary cells, when groundwater elevations in those cells are at an average observed level for the period 1972 to 1983, and the internal gradients are those of the optimal elevations, are shown in Fig. 8. The total regional recharge taking place at the boundary cells for Strategy 3 is 150 Mm<sup>3</sup> (121,000 ac-ft), which is less than 3% higher than that for Strategy 2. However, as shown in Fig. 9, wherever the recharge percentages are greater than 100, the recharge constraints were tight for Strategy 2.

### SUMMARY

A sustained-yield groundwater management scheme based on preassigned 'target' groundwater potentiometric levels is presented. Target levels, especially when selected based on socio-economic considerations, may not represent a physically feasible static surface. A method is presented to find a set of realistic steady-state levels such that the regional sum of its locally weighted deviations from the target set is at a minimum. A multi-objective goal-programming approach is used in which each local groundwater target elevation in a finite-difference grid square is considered an individual objective. A steady-state groundwater flow equation and aquifer physical properties are imposed as constraints and bounds in the optimization process to assure that the developed set of static potentiometric elevations is physically realistic. The steady-state

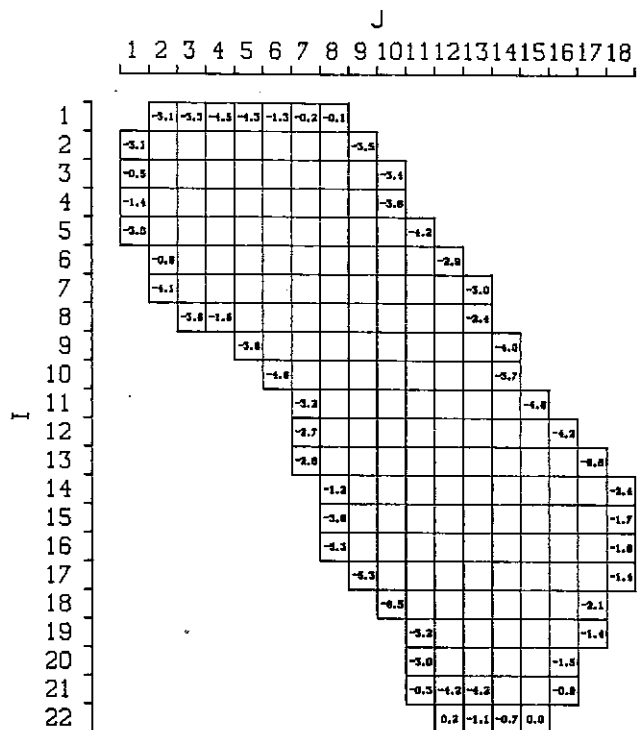


Fig. 8—Recharges required to support optimal elevations of Strategy 3, (Mm<sup>3</sup>).

groundwater withdrawal rates associated with the optimized set of elevations, distributed in finite-difference grid squares, represent a realistic sustained-yield strategy as long as the recharge rates specified in the solution are physically available.

Three sustained-yield pumping strategies are developed for the Grand Prairie of Arkansas, using the management model with two sets of arbitrarily chosen 'target' elevations. The target elevations were created by

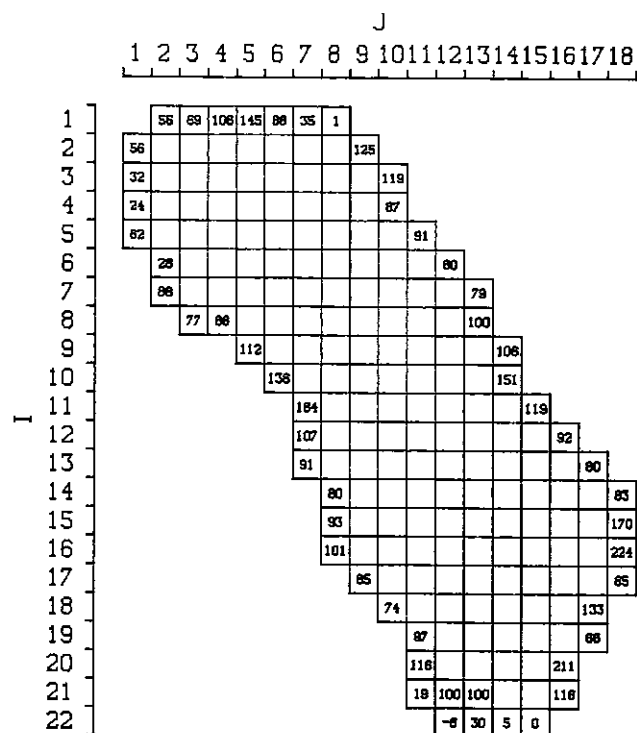


Fig. 9—Recharges required to support Strategy 3 in percent of upper bounds on recharge imposed for development of Strategy 2.

statistically interpolating (kriging) groundwater elevations observed in the Grand Prairie in the springs of 1982 and 1983. These strategies demonstrate a few of the versatile features of the management model. Those features include the following:

1. A more or less uniform sustained-yield pumping strategy can be developed by changing the lower bounds on pumping in internal cells to conform to social or legal criteria;
2. Optimal groundwater elevations can be forced to be comparatively closer to the targets wherever needed by giving higher relative weights to the corresponding difference terms in the objective function;
3. Strategies can be developed in which specific elevations and gradients in a subarea, or specific pumping rate in a cell, are exactly achieved within a regional plan. Such local management objectives, for example, to assure adequate saturated thickness for pumping during a droughty season, can be designed by judiciously using either weighting or tight bounds on acceptable optimal water levels;
4. If the constraints on recharge to the boundary cells are totally relaxed when the optimization is performed for a particular desired potentiometric surface, the procedure calculates the specific recharges (or discharges) in the boundary cells needed to maintain that surface.

The goal-programming concept of minimizing the sum of differences between target and achieved values is formulated in two separate models. One model uses a linear objective function and needs both equality and inequality constraints. The second model employs a quadratic objective function subjected only to inequality constraints. For a groundwater system divided into (n) total grid squares consisting of (m) variable-head cells and (n-m) constant-head cells, each model needs the following number of variables and constraints:

|           | Variables | Constraints |
|-----------|-----------|-------------|
| Linear    | $3m+n$    | $m+n$       |
| Quadratic | $m+n$     | $n$         |

For sample problems consisting of up to 90 finite-difference grid points, the linear program showed significant superiority with regard to computer CPU time-efficiency, although it needs more computer storage than the quadratic model. The linear program, however, did not produce acceptable results in application to the Grand Prairie study area. Due to numerical difficulties encountered in that application, some of the 152 involved equality constraints were violated by the 'optimal' solution. This problem with equality constraints has been experienced by other researchers, and may limit the usefulness of the linear approach.

On the other hand, the quadratic model has been successfully applied to larger regions. These include the 204-cell Grand Prairie as well as a 376-cell region.

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